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Influence Functions for the Infinite and Semi-Infinite Strip

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Influence functions appropriate for the boundary-element method for the Laplace equation are given for the infinite and semi-infinite strip. The method of Green's functions is used to produce single-sum series for the influence functions, which reflect the domain shape and the boundary conditions. Boundary conditions of type 1, 2, and 3 are treated. Series convergence is improved by identifying slowly converging terms and replacing them with fully summed polynomial forms. Numerical examples are given.

Nomenclature

a, b, c, d	=	lengths, m
f	=	specified boundary condition
G	=	Green's function, no units
g	=	internal energy generation, W/m^2
h	=	heat transfer coefficient, W/(m^{2}K)
k	=	thermal conductivity, W/(mK)
N_y	=	norm associated with Y_n, m
n_i	=	outward normal on suface i
P_n	=	kernel function [Eq. (15)], m
P_0	=	kernel function, Table 3, m
S^+, S^-	=	coefficients in Eqs. (15) and (28)
s_i	=	ith surface of domain
T	=	temperature, K
T_∞	=	external temperature for type 3 boundary, K
V_n	=	defined in Eq. (29), m^2
W	=	width of strip, m
x, y	=	coordinate, m
x', y'	=	coordinates of heat source, m
Y	=	eigenfunction, no units
γ_n	=	eigenvalue, Table 2, m^{-1}
δ	=	Dirac delta function, m^{-1}
ϕ	=	influence function: type 1, no units; type 2 or 3, m^{2}K/W

Subscripts

i	=	surface i
j	=	surface j
n	=	nth term of the sum

I. Introduction

In boundary-element methods engineering problems are solved in a domain by superposition of influence functions, which individually satisfy the differential equation and collectively satisfy the boundary conditions. A system of algebraic equations is used to determine coefficients for each boundary element. The method applies to a wide variety of physical phenomena, including heat conduction, fluid flow, and electrochemical potential. Several types of influence functions can be used, and traditional boundary elements involve the infinite-domain solution distributed on each boundary element. Typically uniform, linear, or quadratic distributions are used on each element.[1] For influence functions based on the infinite-domain source solution, the entire boundary of the domain of interest must be discretized.

In this paper influence functions are based on domain-specific solutions to the Laplace equation in the semi-infinite and infinite strip, which satisfy homogeneous boundary conditions everywhere except on one boundary element. These influence functions reflect the shape, boundary conditions, and extent of the domain; consequently, they are more complex than traditional influence functions. However, they have the advantage that boundary elements are needed only on that subset of the boundary that is nonhomogeneous. This is particularly important for the infinite and semi-infinite strip because it is not feasible to discretize the entire boundary.

Three types of boundary elements are treated in this paper: type 1, specified temperature on the element; type 2, specified heat flux on the element; and type 3, specified fluid convection. In heat conduction these influence functions are useful for multiple-body problems, for bodies heated over small regions, and for thermal contact problems.

There has been some work with domain-specific boundary element solutions. The first author has worked with domain-specific influence functions in the infinite strip for conjugate heat transfer[2,3] and in a cylindrical geometry for groundwater flow from a well.[4] In each case the boundary-element solution was based on type 2 influence functions found in closed form by integrating the appropriate Green's function along the boundary. Chang and Sze[5] solved an electropotential problem in a rectangular waveguide using domain-specific Green's functions, with boundary elements placed only on one face of the rectangle. Only a few elements were needed for an accurate solution, and because type 2 boundary elements were used the boundary integral equation contained no singularity.

All of the influence functions discussed in this paper have been developed with the method of Green's functions (GF). A good overview of the method of GF is given in several books.[6–9] Beck et al. in their book[10] give extensive tables of GF for heat conduction and diffusion. The GF are organized with a number system for the number of spatial dimensions, the type of coordinate system, and the type of boundary conditions. Most of the book, however, is devoted to transient heat conduction, and few two-dimensional steady GF are given.

Dolgova and Melnikov[11] discuss steady two-dimensional heat conduction in Cartesian and cylindrical coordinates. Fourier series expansions along one coordinate direction are used to produce single-sum series for the GF. Two examples of GF for the semistrip are given. Most importantly, the slowly converging portions of the series for the GF are identified and replaced with closed-form expressions. This approach has been extended and expanded in two recent books by Melnikov[12,13] to improve the numerical convergence

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of GF for a variety of equations, coordinate systems, and geometries. The chapter on potential fields includes sections on the infinite and semi-infinite strip, and several GF are given. Although wide ranging, the improvement of convergence is applied only to GF.

Marshall[14] discusses Laplace equation solutions for a rectangle with Neumann boundary conditions applied to electrochemical cells. Similar to Melnikov, Marshall replaces the slowly converging portions of the GF with closed-form expressions, some of which are constructed from one-dimensional GF. Numerical examples are given for heating over a small region of a specified-flux boundary to represent a small electrode embedded on a surface. Linton[15] presents rapidly converging expressions for the steady two-dimensional GF on the infinite strip constructed from two transient solutions integrated over time; however, the application was for the water-wave problem in two dimensions.

Recently the authors studied heat conduction in the rectangle[16] and published a complete set of single-summation GF for all possible combinations of boundary condition types 1, 2, and 3. The convergence of the series expressions for the GF was improved by identifying the slowly converging portions of the series and replacing them with fully summed forms.

The present paper is an extension of our work with the rectangle to the semi-infinite and infinite strip. The contribution of the present paper is threefold. First, GF for the infinite and semi-infinite strip are given in single-sum form, several of which have not been published before. Second, integrals are carried out in closed form to produce series expressions for influence functions with boundary conditions of types 1, 2, and 3. To our knowledge the boundary element with type 3 boundary conditions does not seem to have been treated. Third, the numerical properties of the series expressions for the influence functions are improved by replacing slowly converging portions of the series with fully summed forms. The work encompasses 36 geometries with different combinations of boundary condition types 1, 2, and 3.

Here is an overview of the remaining sections of the paper. A general solution for the temperature in the infinite and semi-infinite strip is formally stated with the method of Green's functions. The GF are given in the form of series expressions for a variety of boundary conditions. The influence functions are found from the general temperature solution, and methods for improving the series convergence are discussed. Several numerical examples are given.

II. Temperature Problem

In this section a general solution for the temperature is stated in the form of an integral expression with the method of Green's functions. In a subsequent section the influence functions will be defined in terms of the temperature.

The geometry of the infinite strip is shown in Fig. 1a. Consider the steady temperature in the infinite strip caused either by heating at the boundary or by internal energy generation. The boundary-value problem is governed by

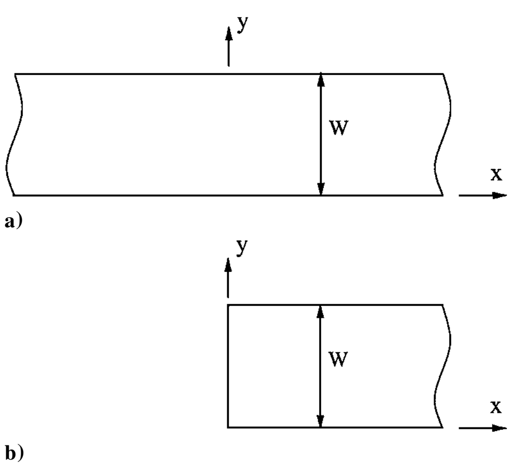

Fig. 1 Geometry of a) infinite strip, $-\infty < x < \infty$ and b) semi-infinite strip, $0 < x < \infty$.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{g(x, y)}{k} \qquad 0 < y < W; -\infty < x < \infty \tag{1}$$

$$k_i \frac{\partial T}{\partial n_i} + h_i T = f_i \qquad \text{at} \qquad y = 0 \qquad \text{or} \qquad y = W \tag{2}$$

$$T, \frac{\partial T}{\partial x} = 0 \qquad \text{as} \qquad x \to \pm\infty \tag{3}$$

Boundary condition (2) represents one of three types at each surface: type 1 for $k_i = 0$, $h_i = 1$, and f_i a specified temperature; type 2 for $k_i = k$, $h_i = 0$, and f_i a specified heat flux; and, type 3 for $k_i = k$ (specified convection condition). For a type 3 boundary heat transfer coefficient h_i must be uniform on that boundary.

The geometry of the semi-infinite strip is shown in Fig. 1b. The temperature in the semi-infinite strip satisfies Eq. (1) on domain $(0 < x < \infty, 0 < y < W)$; however the boundary conditions are given by

$$k_i \frac{\partial T}{\partial n_i} + h_i T = f_i \qquad \text{at} \qquad y = 0, \qquad y = W, \qquad x = 0 \tag{4}$$

$$T, \frac{\partial T}{\partial x} = 0 \qquad \text{as} \qquad x \to \infty \tag{5}$$

The temperature can be formally stated in the form of integrals with the method of Green's functions. For a given geometry, if the Green's function G is known (as defined in the following), then the temperature that satisfies Eq. (1) is given by (see Beck et al.[10], chapter 3)

$$T(x, y) = \int_{x'} \int_{y'=0}^{W} \frac{g(x', y')}{k} G(x, y \mid x', y')\, dx'\, dy'$$

(for volume energy generation)

$$+ \sum_j \int_{s_j} \frac{f_j}{k} G(x, y \mid x'_j, y'_j)\, ds'_j$$

(for boundary conditions of type 2 and 3)

$$- \sum_i \int_{s_i} f_i \frac{\partial G(x, y \mid x'_i, y'_i)}{\partial n'_i}\, ds'_i$$

(for boundary conditions of type 1 only) (6)

The same Green's function appears in each integral but is evaluated at locations appropriate for each integral. Here position (x'_i, y'_i) located on surface s_i and n'_i is the outward facing unit normal on this surface. The summations represent all possible combinations of boundary conditions, but with only one type of boundary on each surface of the strip. Mixed-typed boundary conditions are not treated.

III. Definition of the Green's Function

The steady GF associated with temperature given in Eq. (6) represents the response at point (x, y) caused by a point source of heat located at (x', y'). The GF satisfies the following partial differential equation:

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = -\delta(x - x')\delta(y - y') \tag{7}$$

For the infinite strip, the domain is $(-\infty < x < \infty, 0 < y < W)$, and the boundary conditions are

$$k_i \frac{\partial G}{\partial n_i} + h_i G = 0 \qquad \text{at} \qquad y = 0, \qquad y = W$$

$$G, \frac{\partial G}{\partial x} \text{ are bounded} \qquad \text{as} \qquad x \to \pm\infty \tag{8}$$

For the semi-infinite strip the domain is $(0 < x < \infty, 0 < y < W)$, and the homogeneous boundary conditions are

$$k_i \frac{\partial G}{\partial n_i} + h_i G = 0 \qquad \text{at} \qquad x = 0, \qquad y = 0, \qquad y = W$$

$$G, \frac{\partial G}{\partial x} \text{ are bounded} \qquad \text{as} \qquad x \to +\infty \tag{9}$$

At each finite boundary the boundary condition for G must be of the same type as for the temperature problem.

IV. GF Number

The specific GF and the specific geometry are identified by a "number" of the form XIJYKL in which X and Y represent the coordinate axes, and the letters following each axis name take on numerical values to represent the type of boundary conditions. The value 0 (zero) is used to represent a boundary at infinity. For example, number Y12 represents boundary conditions of type 1 at $y = 0$ and type 2 at $y = W$. As another example, number X10Y13 describes a GF for a semi-infinite strip with a type 1 boundary condition at $x = 0$, a type 1 boundary condition at $y = 0$, and a type 3 boundary condition (convection) at $y = W$. The GF discussed in this paper are designated XI0YKL, which represents 36 different GF for I = 0, 1, 2, or 3 and K, L = 1, 2, or 3. Refer to Beck et al.,[10] chapter 2, for more information on the GF numbering system.

V. Single-Summation Form of the GF

All of the GF discussed in this paper can be stated in the same general form containing a single summation, as follows:

$$G(x, y \mid x', y') = \frac{1}{W} P_0(x, x') + \sum_{n=1}^{\infty} \frac{Y_n(y') Y_n(y)}{N_y(\gamma_n)} P_n(x, x') \tag{10}$$

Eigenfunctions Y_n, norm $N_y^{1/2}$, and kernel function P_n are discussed next. In the preceding equation the summation term is needed for every GF; however, the first term with kernel function P_0 is needed only when Y22 is part of the GF number (i.e., when zero is an eigenvalue).

A. Eigenfunctions and Norms

The y-direction eigenfunction satisfies the following ordinary differential equation:

$$Y_n''(y) + \gamma_n^2 Y_n(y) = 0 \tag{11}$$

(Strictly speaking, the eigenvalues are γ_n^2, which can be shown to be real and nonnegative. Without ambiguity we take the nonnegative square root of γ_n^2 and shall refer to them as the "associated eigenvalues" for brevity.) There are nine different eigenfunctions associated with the nine possible boundary condition combinations YKL (K, L = 1, 2, or 3). Eigenfunctions $Y_n(y)$ are composed of sines and cosines and are given in many texts.[6–10] The norm is defined by

$$N_y(\gamma_n) = \int_0^W Y_n^2(y)\, dy \tag{12}$$

Table 1 contains the eigenfunctions and norms, and Table 2 contains the associated eigenconditions (and eigenvalues for simple cases). For case Y22 the eigenvalue can also take on the value zero, which requires special care.

B. Kernel Functions

The method for obtaining kernel functions P_n will be discussed next. To obtain functions P_n, substitute the series for G given by Eq. (10) into Eq. (7). Additionally the y-portion of the Dirac delta function is replaced with the following identity:

$$\delta(y - y') = \sum_{n=0}^{\infty} \frac{Y_n(y') Y_n(y)}{N_y(\gamma_n)}$$

Table 1 Eigenfunctions and inverse norm[a,b]

Case	$Y_n(y)$	N_y^{-1}
Y11	$\sin(\gamma_n y)$	$2/W$
Y12	$\sin(\gamma_n y)$	$2/W$
Y13	$\sin(\gamma_n y)$	$2\phi_{2n}/W$
Y21	$\cos(\gamma_n y)$	$2/W$
Y22	$\cos(\gamma_n y)$; $\gamma_n \neq 0$	$2/W$ for $\gamma_n \neq 0$
	1; $\gamma_n = 0$	$1/W$ for $\gamma_n = 0$
Y23	$\cos(\gamma_n y)$	$2\phi_{2n}/W$
Y31	$\gamma_n W \cos(\gamma_n y) + (h_1 W/k)\sin(\gamma_n y)$	$2\phi_{1n}/W$
Y32	$\gamma_n W \cos(\gamma_n y) + (h_1 W/k)\sin(\gamma_n y)$	$2\phi_{1n}/W$
Y33	$\gamma_n W \cos(\gamma_n y) + (h_1 W/k)\sin(\gamma_n y)$	$2\Phi_n/W$

[a]Index $n = 1, 2, \ldots$ for all cases except Y22 with $n = 0, 1, 2, \ldots$.
[b]$\phi_{in} = [(\gamma_n W)^2 + (h_i W/k)^2] \div [(\gamma_n W)^2 + (h_i W/k)^2 + h_i W/k]$, $\Phi_n = \phi_{2n} \div [(\gamma_n W)^2 + (h_1 W/k)^2 + (h_1 W/k)\phi_{2n}]$.

Table 2 Eigencondition and eigenvalues for $Y_n(y)$[a]

Case	Eigencondition	Eigenvalues
Y11	$\sin(\gamma_n W) = 0$	$n\pi/W, n = 1, 2, \ldots$
Y12	$\cos(\gamma_n W) = 0$	$(2n-1)\pi/2W, n = 1, 2, \ldots$
Y13	$\gamma_n W \cot(\gamma_n W) = -h_2 W/k$	——
Y21	$\cos(\gamma_n W) = 0$	$(2n-1)\pi/2W, n = 1, 2, \ldots$
Y22	$\sin(\gamma_n W) = 0$	$n\pi/W, n = 0, 1, 2, \ldots$
Y23	$\gamma_n W \tan(\gamma_n W) = h_2 W/k$	——
Y31	$\gamma_n W \cot(\gamma_n W) = -h_1 W/k$	——
Y32	$\gamma_n W \tan(\gamma_n W) = h_1 W/k$	——
Y33	$\tan(\gamma_n W) = [\gamma_n (h_1 + h_2)/k]/[\gamma_n^2 - h_1 h_2 k^{-2}]$	——

[a]Index $n = 1, 2, \ldots$ for all cases except Y22 with $n = 0, 1, 2, \ldots$.

It is very important to include the $n = 0$ term of the summation only for case Y22. Then Eq. (10) can be written

$$\sum_{n=0}^{\infty} \frac{Y_n(y') Y_n(y)}{N_y(\gamma_n)} \left[\frac{d^2 P_n}{dx^2} - \gamma_n^2 P_n + \delta(x - x') \right] = 0 \tag{13}$$

This equation is satisfied if the term in brackets is zero for all values of n. That is, function P satisfies the following ordinary differential equation:

$$\frac{d^2 P_n}{dx^2} - \gamma_n^2 P_n = -\delta(x - x') \tag{14}$$

The solution for P_n can be found using two solutions of the homogeneous equation that satisfy the boundary conditions and are joined appropriately at $x = x'$ (see, for example, Stakgold,[8] chapter 1). Typically the kernel functions are expressed as hyperbolic trigonometric functions. However we find that a better form involves exponential functions with negative arguments, which when evaluated never cause numerical overflow for large x. Further, this allows the kernel functions to be stated in a single formula that is convenient for computer evaluation (for $n \neq 0$):

$$P_n(x, x') = \frac{1}{2\gamma_n S^+} \left[S^+ \exp(-\gamma_n |x - x'|) + S^- \exp(-\gamma_n |x + x'|) \right] \tag{15}$$

where the values for S^+ and S^- are as follows: case X00, $S^+ = 1$ and $S^- = 0$; case X10, $S^+ = 1$ and $S^- = -1$; case X20, $S^+ = S^- = 1$; and, case X30, $S^+ = \gamma_n W + hW/k$ and $S^- = \gamma_n W - hW/k$.

C. Kernel Functions for $\gamma_n = 0$

There are four geometries for which a zero eigenvalue must be included: cases XI0Y22 for I = 0, 1, 2, and 3. In these cases the kernel function P_0 for cases XI0 are found from

$$\frac{d^2 P_0}{dx^2} = -\delta(x - x') \tag{16}$$

For cases X10Y22 and X30Y22 function P_0 can be found by using two solutions to the differential equation that are matched

Table 3 Kernel function $P_0(x, x')$

Case	$P_0(x, x')$
X00	$-\frac{1}{2}\lvert x - x'\rvert$
X10	$-\frac{1}{2}\lvert x - x'\rvert + \frac{1}{2}\lvert x + x'\rvert$
X20	$-\frac{1}{2}\lvert x - x'\rvert - \frac{1}{2}\lvert x + x'\rvert$
X30	$-\frac{1}{2}\lvert x - x'\rvert + \frac{1}{2}\lvert x + x'\rvert + k/h$

appropriately at $x = x'$. Special cases X00Y22 and X20Y22 are treated in the next section.

D. Special Cases X20Y22 and X00Y22

In the definition of the GF, the boundedness of G and $\partial G/\partial x$ as $|x| \to \infty$ imply that kernel function P_0 along with $\partial P_0/\partial x$ are bounded as $|x| \to \infty$. For cases X20Y22 and X00Y22 such a P_0 does not exist, and it is necessary to modify the definition for the GF. Barton[9] considered these cases and removed the boundedness condition on P_0 as $|x| \to \infty$; this leads to unbounded solutions. For case X20Y22 the modified P_0 function is determined from two solutions to Eq. (16) that satisfy the boundary condition at $x = 0$, a symmetry condition in x and x', and two matching conditions at $x = x'$. For case X00Y22 a "radiation" condition is imposed, which requires the modified P_0 function to depend on x and x' only through $|x - x'|$.

The modified Green's function, denoted G_M, is unique only up to an additive constant. The series form of G_M is the same as given by Eq. (10). Solutions for $P_n(x, x')$ for $n \neq 0$ are again given by Eq. (15). Function G_M differs from the regular Green's function G already discussed only in function $P_0(x, x')$, which is unbounded as $|x| \to \infty$. The four P_0 functions for cases XI0Y22 (I = 0, 1, 2, and 3) are given in Table 3.

In cases X20Y22 and X00Y22 additional constraints are needed in order for temperature solutions to exist. The input data to the temperature problem must satisfy an energy balance: the sum of the heat passing through the boundaries of the body must be equal to the (negative of the) integral of the heat introduced by volume energy generation. If the volume energy generation is zero, then the boundary heat fluxes must sum to zero. Physically, this constraint is needed because there is no global "heat sink" to which the introduced heat can flow (because there are no type 1 or type 3 boundary conditions present).

To find the temperature with the modified GF, we have, similar to Eq. (6), the following:

$$T(x, y) = \sum \int_{s_i} \frac{f_i}{k} G_M(x, y \mid x_i', y_i')\, ds_i' + \iint \frac{g}{k} G_M(x, y \mid x', y')\, dx'\, dy' \tag{17}$$

provided that T and $\partial T/\partial x$ tend to zero as $|x| \to \infty$ in such a way that G_M and $\partial G_M/\partial x$ also tend to zero as $|x| \to \infty$. We note that the arbitrary additive constant in G_M does contribute to $T(x, y)$ when the energy balance between f_i and g is imposed.

VI. Influence Functions

The influence function is the response of the body to unit heating over a single boundary element; elsewhere the boundaries are homogeneous. If $\phi(x, y)$ denotes the influence function, then the temperature caused by that boundary element is given by $T(x, y) = \phi(x, y) f$, where f is the surface element strength. The temperature caused by all of the boundary elements on a body is the sum of all individual effects, that is, $T(x, y) = \sum_i \phi_i(x, y) f_i$. In this section series expressions for the influence function are given; there are different expressions depending on which surface the boundary element is located and on the boundary element type (1, 2, or 3).

A. Element on $y = 0$ Infinite Boundary

Consider an infinite or semi-infinite strip with a boundary element located at $a < x < b$ on the infinitely long boundary at $y = 0$. The boundary heating associated with this element is given by

$$f(x) = \begin{cases} 1; & a < x < b \\ 0; & \text{otherwise} \end{cases} \tag{18}$$

The other boundaries are homogeneous ($f_i = 0$), but they can be of any type (1, 2, or 3).

Boundary Element of Type 2 or 3

For a boundary element of type 2 or 3, the influence function is found by applying unit heating from Eq. (18) in the second integral term from Eq. (6):

$$\phi(x, y) = \frac{1}{k} \int_a^b G(x, y, \mid x', y' = 0)\, dx' \tag{19}$$

For a type 2 element the element strength f is heat flux, and heat flux is the coefficient associated with that boundary element. For a type 3 element the element strength is $f = hT_\infty$. External temperature T_∞ is the coefficient associated with that boundary element, and h must be a uniform value over the entire $y = 0$ boundary. Type 3 boundary elements are useful for contact conductance problems.

When the integral in the preceding expression is combined with the single-sum series for the GF [Eq. (10)], the influence function is given by

$$\phi(x, y) = \frac{1}{kW} \int_a^b P_0(x, x')\, dx' + \sum_{n=1}^{\infty} \frac{Y_n(y) Y_n(y' = 0)}{k\, N_y(\gamma_n)} \int_a^b P_n(x, x')\, dx' \tag{20}$$

Integrals of the kernel functions are needed; integrals of P_0 are given in Table 4, and integrals of P_n are given in Table 5. The preceding expression can also be used to construct boundary elements at $y = W$ by reversing the y-coordinate system.

Boundary Element of Type 1

For a boundary element of type 1, the influence function is given by substituting the unit heating from Eq. (18) into the third integral term in Eq. (6):

$$\phi(x, y) = \int_a^b \left.\frac{\partial G}{\partial y'}\right|_{y'=0} dx' \tag{21}$$

The sign is positive because n_i' and y' are in opposite directions. Substitute the series form of the GF [Eq. (10)] to obtain

$$\phi(x, y) = \sum_{n=1}^{\infty} \frac{Y_n(y)}{N_y(\gamma_n)} \left.\frac{dY_n}{dy'}\right|_{y'=0} \int_a^b P_n(x, x')\, dx' \tag{22}$$

Table 4 Integral of kernel function $P_0(x, x')$ over (a, b)

Case	Range of x	$\int_a^b P_0(x, x')\, dx'$
X00	$x < a$	$-\frac{1}{4}(b^2 - a^2) + \frac{1}{2}x(b - a)$
	$a < x < b$	$\frac{1}{2}x^2 - \frac{1}{4}(b^2 + a^2) + \frac{1}{2}x(b - a)$
	$b < x$	$-\frac{1}{4}(b^2 - a^2) - \frac{1}{2}x(b - a)$
X10	$x < a$	$x(b - a)$
	$a < x < b$	$-\frac{1}{2}(x^2 + a^2) + xb$
	$b < x$	$\frac{1}{2}(b^2 - a^2)$
X20	$x < a$	$\frac{1}{2}(b^2 - a^2)$
	$a < x < b$	$-\frac{1}{2}(x^2 + b^2) + xa$
	$b < x$	$-x(b - a)$
X30	$x < a$	$(x + k/h)(b - a)$
	$a < x < b$	$-\frac{1}{2}(x^2 + a^2) + (b - a)k/h + xb$
	$b < x$	$\frac{1}{2}(b^2 + a^2) + (b - a)k/h$

Table 5 Integral of kernel function $P_n(x, x')$ over (a, b)

Range of x	$\int_a^b P_n(x, x')\,dx'$ [a]
$x < a$	$(F + S^+\{\exp[-\gamma_n(a-x)] - \exp[-\gamma_n(b-x)]\})/(2\gamma_n^2 S^+)$
$a < x < b$	$(F - S^+\{\exp[-\gamma_n(b-x)] + \exp[-\gamma_n(x-a)]\})/(2\gamma_n^2 S^+) + 1/\gamma_n^2$
$x > b$	$(F + S^+\{\exp[-\gamma_n(x-b)] - \exp[-\gamma_n(x-a)]\})/(2\gamma_n^2 S^+)$

[a] $F = S^-\{\exp[-\gamma_n(x+a)] - \exp[-\gamma_n(x+b)]\}$.

Table 6 Derivative of eigenvalue dY_n/dy'

Case ($J = 1, 2, 3$)	dY_n/dy'	$dY_n/dy'\|_{y'=0}$
Y1J	$\gamma_n \cos\gamma_n y'$	γ_n
Y2J	$-\gamma_n \sin\gamma_n y'$	0
Y3J	$-\gamma_n^2 W \sin\gamma_n y' + \gamma_n B_1 \cos\gamma_n y'$	$\gamma_n B_1$ [a]

[a] $B_1 = h_1 W/k$.

Table 7 Integral of eigenfunction $Y_n(y')$ over (c, d)

Case ($J = 1, 2, 3$)	$\int_c^d Y_n(y')\,dy'$
Y1J	$[-\cos\gamma_n d + \cos\gamma_n c]/\gamma_n$
Y2J	$[-\sin\gamma_n d + \sin\gamma_n c]/\gamma_n$ (note $\gamma_n \neq 0$)
Y3J	$W[\sin\gamma_n d - \sin\gamma_n c] + B_1$[a]$[-\cos\gamma_n d + \cos\gamma_n c]/\gamma_n$

[a] $B_1 = h_1 W/k$.

Again the integrals of the kernel function are needed, and these are given in Tables 4 and 5. The derivative of the eigenfunction needed for the preceding expression is given in Table 6.

The influence functions just discussed are for boundary elements located on the infinitely long boundary of the strip. In the next section influence functions are discussed for the semi-infinite strip for heating along the finite-length boundary at $x = 0$.

B. Element on $x = 0$ Finite Boundary

Consider the semi-infinite strip with a boundary element located over $c < y < d$ on the finite-length boundary at $x = 0$. The boundary heating function associated with this boundary element is given by

$$f(y) = \begin{cases} 1; & c < y < d \\ 0; & \text{otherwise} \end{cases} \tag{23}$$

As before, the other boundary conditions (at $y = 0$ and W) are homogeneous but can be of any type (1, 2, or 3).

Element of Type 2 or 3

For a boundary element of type 2 or 3, the influence function is given by substituting Eq. (23) into the first integral term of Eq. (6):

$$\phi(x, y) = \frac{1}{k}\int_c^d G(x, y, \mid x' = 0, y')\,dy' \tag{24}$$

Upon substitution of the series expression for G [Eq. (10)], the influence function can be written as

$$\phi(x, y) = \frac{(d-c)}{kW} P_0(x, x' = 0) + \sum_{n=1}^{\infty} \frac{Y_n(y)}{k\,N_y(\gamma_n)} P_n(x, x' = 0)\int_c^d Y_n(y')\,dy' \tag{25}$$

Here the integrals of eigenfunction Y_n are needed; these are given in Table 7.

Element of Type 1

For a boundary element of type 1, the influence function for heating over surface $x = 0$ is given by substituting Eq. (23) into the third integral term of Eq. (6):

$$\phi(x, y) = \int_c^d \left.\frac{\partial G}{\partial x'}\right|_{x'=0} dy' \tag{26}$$

Upon substitution of the series expression for G, the influence function is

$$\phi(x, y) = \sum_{n=1}^{\infty} \frac{Y_n(y)}{N_y(\gamma_n)} \left.\frac{\partial P_n(x, x')}{\partial x'}\right|_{x'=0} \int_c^d Y_n(y')\,dy' \tag{27}$$

As before, integrals of Y_n are listed in Table 7. The derivative of the kernel function P_n evaluated at $x' = 0$ is given by

$$\left.\frac{\partial P_n(x, x')}{\partial x'}\right|_{x'=0} = \frac{1}{2S^+}\left[S^+\exp(-\gamma_n x) - S^-\exp(-\gamma_n x)\right] \tag{28}$$

VII. Improvement of Convergence

Whenever infinite series are part of a numerical method, the series' convergence behavior is of central importance. Generally influence functions whose series contain exponential functions converge rapidly because the exponential terms rapidly go to zero as series index n increases. However, some of the influence functions just discussed converge very slowly. In this section the slowly converging terms are identified and replaced by fully summed forms to greatly improve the convergence of these series.

A. Element on $y = 0$ Infinite Boundary

When the boundary element is located on infinitely long surface at $y = 0$, suppose the influence function is located on $a < x < b$. In this case the integral of the kernel function P_n contains the additive term $1/\gamma_n^2$. It is this additive term that appears only for $a < x < b$ which causes poor series convergence. To identify how this term causes convergence problems, write the integral of the kernel function in two parts:

$$\int_a^b P_n(x, x')\,dx' = V_n(x, a, b) + \frac{1}{\gamma_n^2} \tag{29}$$

where function $V_n(x, a, b)$ contains exponential terms from Table 5 (for $a < x < b$ only). The expression for the influence function depends on the type of boundary condition.

Element of Type 2 or 3

For a boundary element of type 2 or 3, substitute the preceding expression into Eq. (20) to obtain (for $a < x < b$ only):

$$\phi(x, y) = \frac{1}{W}\int_a^b P_0(x, x')\,dx' + \sum_{n=1}^{\infty} \frac{Y_n(y) Y_n(y' = 0)}{N_y(\gamma_n)} V_n(x, a, b) + \sum_{n=1}^{\infty} \frac{Y_n(y) Y_n(0)}{N_y(\gamma_n)} \frac{1}{\gamma_n^2} \tag{30}$$

The slowly converging term is the last series in the preceding expression. As n increases, the terms in this series diminish approximately as $1/n^2$, and thousands of terms may be needed for accurate evaluation. However, this series is identical to the one-dimensional temperature in a rectangular slab body heated at $y = 0$ and with a homogeneous condition at $y = W$. This one-dimensional temperature series may be replaced by a fully summed polynomial form given in Table 8b for several combinations of boundary condition types;

Table 8a One-dimensional temperature caused by heating at a type 1 boundary condition at $y = 0$. The other boundary condition is homogeneous

Case	$T_{1D}(y)/T_0$
Y11	$1 - y/W$
Y12	1
Y13	$1 - [B_2/(1 + B_2)]y/W$

Table 8b One-dimensional temperature caused by heating at a boundary condition of type 2($f = q$) or type 3 ($f = h_1 T_\infty$) located at $y = 0$. The boundary condition at $y = W$ is homogeneous. Here $B_1 = h_1 W/k$ and $B_2 = h_2 W/k$

Case	$T_{1D}(y)/(f\ W/k)$
Y21	$1 - y/W$
Y22	$\frac{1}{2}(y/W)^2 - y/W + \frac{1}{3}$
Y23	$1 + 1/B_2 - y/W$
Y31	$(1 - y/W)/(1 + B_1)$
Y32	$1/B_1$
Y33	$(1 - B_2 y/W + B_2)/(B_1 + B_2 + B_1 B_2)$

a complete discussion of one-dimensional temperatures is given in our previous paper.[16]

Element of Type 1

For a boundary element of type 1, substitute Eq. (29) into Eq. (22) to obtain (for $a < x < b$ only)

$$\phi(x, y) = \sum_{n=1}^{\infty} \frac{Y_n(y)}{N_y(\gamma_n)} \left.\frac{\mathrm{d}Y_n}{\mathrm{d}y'}\right|_{y'=0} V_n(x, a, b)$$

$$+ \sum_{n=1}^{\infty} \frac{Y_n(y)}{N_y(\gamma_n)} \left.\frac{\mathrm{d}Y_n}{\mathrm{d}y'}\right|_{y'=0} \frac{1}{\gamma_n^2} \qquad (31)$$

In this expression the first series contains exponential terms in V_n and converges rapidly. The second series converges slowly, and it is identical to the one-dimensional temperature in a rectangular slab body with a specified temperature at $y = 0$ and with a homogeneous boundary condition at $y = W$. As before, this one-dimensional temperature series can be replaced by a fully summed polynomial form given in Table 8a. For type 1 elements the improved convergence series is especially important if the heat flux is needed. Without the improvement in convergence, the series for the heat flux actually diverges on the type 1 element.

B. Element on $x = 0$ Finite Boundary

On the semi-infinite strip, when a boundary element is located on finite surface $x = 0$, the already discussed improvement of convergence is not possible, as the integral of the kernel function does not appear. If a better-converging form is needed, there are two ways to proceed. First, an approximate influence function for the semi-infinite strip can be constructed from a rectangle with a large aspect ratio. For the rectangle the kernel functions can be arranged along the y direction, and then the slowly converging terms can be identified and replaced as before. See our previous paper[16] for a discussion of the rectangle. Second, another form of the exact influence function with very good convergence behavior can be constructed from two forms of the transient GF by the method of time partitioning. Details of the method and the required transient GF are given by Beck et al.,[10] chapter 5.

VIII. Numerical Examples

In this section numerical values are given from several specific influence functions. The numerical results were carried out in Fortran 77 compiled on a DEC Alpha computer. The convergence of the series was determined by computing the sum of the last five terms of the series and comparing it to the partial sum. The series was truncated when the sum of the (absolute value of) last five terms divided by the partial sum was less than 10^{-6}.

Table 9 Normalized influence function values, infinite strip, for a type 1 boundary element located at ($y = 0, -b < x < b$)

Case	x/b	y/W	ϕ	N
X00Y11	0.0500	0.0000	1.0000000	5
	0.0500	0.1111	0.8324688	10
	0.0500	0.5555	0.3199460	10
	0.5500	0.0000	1.0000000	5
	0.5500	0.1111	0.7815413	20
	0.5500	0.5555	0.2835486	20
	0.7500	0.0000	1.0000000	5
	0.7500	0.1111	0.7021008	30
	0.7500	0.5555	0.2537367	30

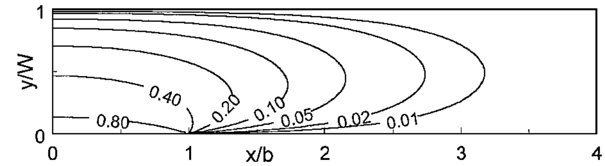

Fig. 2 Contours of the influence function for a type 1 boundary element located at ($-b < x < b$) on the $y = 0$ surface of the infinite strip of width $W = 2b$. The $y = W$ surface is at zero temperature. Values are normalized as T/T_0.

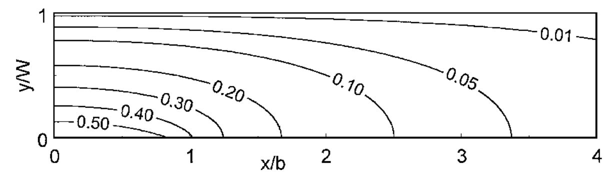

Fig. 3 Contours of the influence function for a type 2 boundary element for the same conditions as Fig. 1. Values are normalized as $T/(q_0 W/k)$.

A. Case X00Y11

Results for a type 1 element (specified temperature) on the $y = 0$ surface of the infinite slab have been computed from Eq. (31). The thickness is $W = 2b$, and the unheated surface of the slab is held at zero temperature (type 1 boundary condition). The surface element is located over $-b < x < b$. The designation for this case is X00Y11. In Fig. 2 contours of normalized temperature are plotted in the region $0 < x < 4$, and region $x < 0$ can be inferred by symmetry. The temperature is normalized as T/T_0, where T_0 is the temperature on the boundary element. At $y = 0$ the normalized temperature is discontinuous at $x/b = 1$, with value unity on the boundary element and zero off the element. Inside the body the temperature varies smoothly, falling rapidly toward zero for $x > 1$. Numerical values for the influence function are also given in Table 9 along with the number of series terms needed to evaluate them.

B. Case X00Y21

Results for a type 2 boundary element (specified heat flux) on the $y = 0$ surface of the infinite slab have been computed from Eq. (30). The thickness is $W = 2b$, and the unheated surface of the slab is held at zero temperature (type 1 boundary condition at $y = W$). The surface element is located over $-b < x < b$. The designation for this case is X00Y21, and a contour plot is shown in Fig. 3. The temperature is normalized as $T/(q_0 W/k)$, where q_0 is the heat flux on the boundary element. On the $y = 0$ surface the temperature is continuous across the edge of the boundary element (at $x/b = 1$) even though the heat flux is discontinuous there. The boundary heat flux is proportional to the slope of the contour lines. For example, the contours that are perpendicular to the $y = 0$ surface for $x/b > 1$ indicate the zero-flux conditions there. Some numerical values for this example are given in Table 10.

Table 10 Normalized influence function values, infinite strip, for boundary element located at ($y = 0, -b < x < b$)

Case	x/b	y/W	$\phi \cdot k/W$	N^a
X00Y21	0.0500	0.0000	0.6208179	10
	0.0500	0.1111	0.5167667	5
	0.0500	0.5555	0.2143039	5
	0.5500	0.0000	0.5748256	15
	0.5500	0.1111	0.4738847	5
	0.5500	0.5555	0.1991702	5
	0.7500	0.0000	0.5289062	20
	0.7500	0.1111	0.4330441	5
	0.7500	0.5555	0.1867629	5
X00Y31	0.0500	0.0000	0.5808279	10
(Biot = 0.1)	0.0500	0.1111	0.4828489	5
	0.0500	0.5555	0.1993064	5
	0.5500	0.0000	0.5373122	15
	0.5500	0.1111	0.4420167	15
	0.5500	0.5555	0.1848974	15
	0.7500	0.0000	0.4935517	20
	0.7500	0.1111	0.4032389	20
	0.7500	0.5555	0.1732564	20
X00Y31	0.0500	0.0000	0.3740024	10
(Biot = 1.0)	0.0500	0.1111	0.3088349	10
	0.0500	0.5555	0.1240705	10
	0.5500	0.0000	0.3454309	15
	0.5500	0.1111	0.2801587	15
	0.5500	0.5555	0.1136545	15
	0.7500	0.0000	0.3149554	20
	0.7500	0.1111	0.2517529	20
	0.7500	0.5555	0.1052628	20

[a]Quantity N is the number of series terms needed.

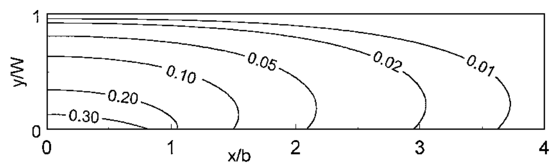

Fig. 4 Contours of the influence function for a type 3 boundary element for the same conditions as Fig. 1 and with $hW/k = 1.0$. Values are normalized as $T/(hT_\infty W/k)$.

C. Case X00Y31

Results for a type 3 boundary element (specified convection) on the infinite slab are discussed in this example. As before, the thickness is $W = 2b$, and the unheated surface of the slab is held at zero temperature (type 1 boundary condition at $y = W$). The surface element is located at $y = 0$ over $-b < x < b$. The designation for this case is X00Y31. For a type 3 boundary element the heat-transfer coefficient is a constant over the entire boundary, and it is the convective temperature T_∞ that is nonzero on the element and zero elsewhere. In Fig. 4 contours of the normalized temperature at $y = 0$ are shown for $hW/k = 1.0$. The temperature is normalized as $T/(hT_\infty W/k)$. At the $y = 0$ boundary the slope of the contour lines indicate the direction of heat flow. Contour lines with negative slope over the boundary element ($0 < x/b < 1$) indicate heat flux entering the domain, and contour lines with positive slope outside of the boundary element indicate heat leaving the domain. If the Biot number (hW/k) is varied, the type 3 element can approach the behavior of a type 1 element (for Biot number $\rightarrow \infty$) or a type 2 element (for Biot number $\rightarrow 0$). Numerical values for this case are shown in Table 10, along with values for the same geometry but with Biot = 0.1. The eigenvalues for case X00Y31 were computed by the method of Haji-Shiekh and Beck.[17]

D. Case X20Y11

Results for a type 2 boundary element on the $x = 0$ face of the semistrip ($0 < x < \infty$, $0 < y < W$) are discussed in this example. The boundaries at $y = 0$ and W are held at zero temperature. The type 2 boundary element (specified flux) is located over ($0.5 < y/W < 0.8$), and the remainder of the $x = 0$ boundary is insulated. A contour plot of the normalized temperature $T/(q_0 W/k)$ is shown in Fig. 5. The y axis is stretched relative to the x axis in order to include more contour lines. The temperature is highly localized to the boundary element and quickly decays toward zero away from this region. The peak value occurs in the center of the boundary element. Table 11 gives several numerical values from this example along with the number of series terms needed. A few more terms are needed on the $x = 0$ boundary.

Table 11 Influence-function values for the semistrip with a type 2 boundary element at ($x = 0, 0.5 < y/W < 0.8$)

Case	x/W	y/W	$\phi \cdot k/W$	N^a
X20Y11	0.0000	0.0000	0.0000000	5
	0.0000	0.1111	0.0212925	105
	0.0000	0.5555	0.2076603	115
	0.0500	0.0000	0.0000000	5
	0.0500	0.1111	0.0210852	40
	0.0500	0.5555	0.1659574	35
	0.5500	0.0000	0.0000000	5
	0.5500	0.1111	0.0086034	10
	0.5500	0.5555	0.0294469	10
	0.7500	0.0000	0.0000000	5
	0.7500	0.1111	0.0049281	10
	0.7500	0.5555	0.0155123	10

[a]Quantity N is the number of series terms needed.

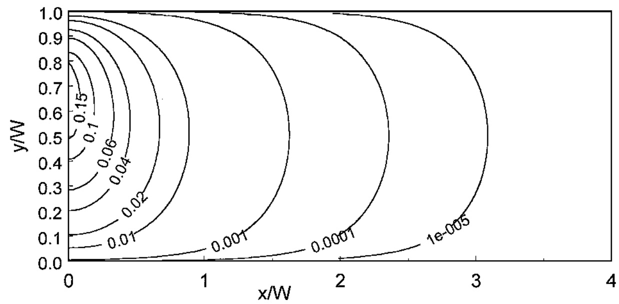

Fig. 5 Contours of the influence function for a type 2 boundary element located at ($0.5 < y/W < 0.8$) on the $x = 0$ surface of the semi-infinite strip. The surfaces at $y = 0$ and W are at zero temperature. Values are normalized as $T/(q_0 W/k)$.

IX. Summary

In this paper influence functions appropriate for steady two-dimensional heat conduction are given for the infinite and semi-infinite strip. All combinations of boundary conditions of types 1, 2, and 3 are treated. Series expressions are given with improved numerical convergence and several numerical examples are presented.

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